

# MINIMAL ZERO-SUM SEQUENCES OF LENGTH FOUR OVER CYCLIC GROUP WITH ORDER $n = p^\alpha q^\beta$

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**ABSTRACT.** Let  $G$  be a finite cyclic group. Every sequence  $S$  over  $G$  can be written in the form  $S = (n_1g) \cdots (n_kg)$  where  $g \in G$  and  $n_1, \dots, n_k \in [1, \text{ord}(g)]$ , and the index  $\text{ind}S$  of  $S$  is defined to be the minimum of  $(n_1 + \cdots + n_k)/\text{ord}(g)$  over all possible  $g \in G$  such that  $\langle g \rangle = G$ . A conjecture says that if  $G$  is finite such that  $\gcd(|G|, 6) = 1$ , then  $\text{ind}(S) = 1$  for every minimal zero-sum sequence  $S$ . In this paper, we prove that the conjecture holds if  $|G|$  has two prime factors.

*Key Words:* minimal zero-sum sequence, cyclic groups, index of sequences.

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## 1. INTRODUCTION

Throughout the paper, let  $G$  be an additively written finite cyclic group of order  $|G| = n$ . By a sequence over  $G$  we mean a finite sequence of terms from  $G$  which is unordered and repetition of terms is allowed. We view sequences over  $G$  as elements of the free abelian monoid  $\mathcal{F}(G)$  and use multiplicative notation. Thus a sequence  $S$  of length  $|S| = k$  is written in the form  $S = (n_1g) \cdots (n_kg)$ , where  $n_1, \dots, n_k \in \mathbb{N}$  and  $g \in G$ . We call  $S$  a *zero-sum sequence* if  $\sum_{j=1}^k n_jg = 0$ . If  $S$  is a zero-sum sequence, but no proper nontrivial subsequence of  $S$  has sum zero, then  $S$  is called a *minimal zero-sum sequence*. Recall that the index of a sequence  $S$  over  $G$  is defined as follows.

**Definition 1.1.** For a sequence over  $G$

$$S = (n_1g) \cdots (n_kg), \quad \text{where } 1 \leq n_1, \dots, n_k \leq n,$$

the index of  $S$  is defined by  $\text{ind}(S) = \min\{\|S\|_g \mid g \in G \text{ with } \langle g \rangle = G\}$ , where

$$(1.1) \quad \|S\|_g = \frac{n_1 + \cdots + n_k}{\text{ord}(g)}.$$

Clearly,  $S$  has sum zero if and only if  $\text{ind}(S)$  is an integer.

**Conjecture 1.2.** Let  $G$  be a finite cyclic group such that  $\gcd(|G|, 6) = 1$ . Then every minimal zero-sum sequence  $S$  over  $G$  of length  $|S| = 4$  has  $\text{ind}(S) = 1$ .

The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. It was first addressed by Kleitman-Lemke (in the conjecture [9, page 344]), used as a key tool by Geroldinger ([6, page 736]), and then

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investigated by Gao [3] in a systematical way. Since then it has received a great deal of attention (see for example [1, 2, 4, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18]). A main focus of the investigation of index is to determine minimal zero-sum sequences of index 1. If  $S$  is a minimal zero-sum sequence of length  $|S|$  such that  $|S| \leq 3$  or  $|S| \geq \lfloor \frac{n}{2} \rfloor + 2$ , then  $\text{ind}(S) = 1$  (see [1, 14, 16]). In contrast to that, it was shown that for each  $k$  with  $5 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$ , there is a minimal zero-sum subsequence  $T$  of length  $|T| = k$  with  $\text{ind}(T) \geq 2$  ([13, 15]) and that the same is true for  $k = 4$  and  $\gcd(n, 6) \neq 1$  ([13]). The left case leads to the above conjecture.

In [12], it was proved that Conjecture 1.2 holds true if  $n$  is a prime power. In [11], it was proved that Conjecture 1.2 holds for  $n = p_1^\alpha \cdot p_2^\beta$ , ( $p_1 \neq p_2$ ), and at least one  $n_i$  co-prime to  $|G|$ . However, the general case is still open. In [19], it was proved that Conjecture 1.2 holds if the sequence  $S$  is reduced and at least one  $n_i$  co-prime to  $|G|$ .

In this paper, we give the affirmative proof of Conjecture 1.2 for general case under assumption  $n = p^\alpha q^\beta$ .

**Theorem 1.3.** *Let  $G$  be a finite cyclic group of order  $|G| = p^\alpha q^\beta$ , where  $\alpha, \beta \in \mathbb{N}$ , and  $p, q$  are distinct primes, such that  $\gcd(|G|, 6) = 1$ . Then every minimal zero-sum sequence  $S$  over  $G$  of length  $|S| = 4$  has  $\text{ind}(S) = 1$ .*

It was mentioned in [13] that Conjecture 1.2 was confirmed computationally if  $n \leq 1000$ . Hence, throughout the paper, we always assume that  $n > 1000$ .

## 2. REDUCTION TO A SPECIAL CASE

Given real numbers  $a, b \in \mathbb{R}$ , we use  $[a, b] = \{x \in \mathbb{Z} | a \leq x \leq b\}$  to denote the set of integers between  $a$  and  $b$ , and similarly, we set  $[a, b) = \{x \in \mathbb{Z} | a \leq x < b\}$ . For  $x \in \mathbb{Z}$ , we denote by  $|x|_n \in [1, n]$  the integer congruent to  $x$  modulo  $n$ .

Throughout this paper, let  $G$  be a finite cyclic group of order  $|G| = n = p^\alpha q^\beta > 1000$ , where  $\alpha, \beta \in \mathbb{N}$  and  $p, q$  are distinct primes greater than or equal to 5.

First we show that Theorem 1.3 can be reduced to sequences of a special form.

**Proposition 2.1.** *Let  $S = (eg) \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g)$  be a minimal zero-sum sequence over  $G$ , where  $g \in G$  with order  $\text{ord}(g) = |G| = p^\alpha q^\beta$  and  $e, a, b, c \in [1, n-1]$  such that  $e < a \leq b < c < \frac{n}{2}$  and  $e + c = a + b$ . Then  $\text{ind}(S) = 1$ .*

*Proof.* Proof of Theorem 1.3 based on Proposition 2.1. Let  $S = (n_1g) \cdot (n_2g) \cdot (n_3g) \cdot (n_4g)$  where  $g \in G$  with  $\text{ord}(g) = |G|$  and  $n_1, n_2, n_3, n_4 \in [1, n-1]$ . Now do the reduction to the special case in Proposition 2.1.

Notice the following two sufficient conditions (introduced in Remark 2.1 of [11]):

(1) If there exists positive integer  $m$  such that  $\gcd(n, m) = 1$  and  $|mn_1|_n + |mn_2|_n + |mn_3|_n + |mn_4|_n = 3n$ , then  $\text{ind}(S) = 1$ .

(2) If there exists positive integer  $m$  such that  $\gcd(n, m) = 1$  and at most one  $|mn_i|_n \in [1, \frac{n}{2}]$  (or, similarly, at most one  $|mn_i|_n \in [\frac{n}{2}, n]$ ), then  $\text{ind}(S) = 1$ .

Hence we can assume that  $n_1 + n_2 + n_3 + n_4 = 2n$  and  $n_1 \leq n_2 < \frac{n}{2} < n_3 \leq n_4$ . By the minimality of  $S$ , it doesn't hold  $n_1 + n_4 = n$ . Next we may assume that  $n_1 + n_4 < n$ . Otherwise

we let  $m = n - 1$  and consider the sequence

$$(n'_1, n'_2, n'_3, n'_4) = (|mn_4|_n, |mn_3|_n, |mn_2|_n, |mn_1|_n) = (n - n_4, n - n_3, n - n_2, n - n_1).$$

Let  $e = n_1, c = n_2, b = n - n_3$  and  $a = n - n_4$ , then  $e < a \leq b < c < \frac{n}{2}$  and  $n_1 + n_2 + n_3 + n_4 = 2n$  implies that  $e + c = a + b$ .  $\square$

Proposition 2.1 is already well-known in some special cases. The following three lemmas are analogues of Lemma 2.3, Lemma 2.5 and Lemma 2.6 in [11], and the proof is very similar.

**Lemma 2.2.** *Proposition 2.1 holds if one of the following conditions holds :*

(1) *There exist positive integers  $k, m$  such that  $\frac{kn}{c} \leq m \leq \frac{kn}{b}$ ,  $\gcd(m, n) = 1$ ,  $1 \leq k \leq b$  and  $ma < n$ .*

(2) *There exists a positive integer  $M \in [1, \frac{n}{2e}]$  such that  $\gcd(M, n) = 1$  and at least two of the following inequalities hold :*

$$|Ma|_n > \frac{n}{2}, |Mb|_n > \frac{n}{2}, |Mc|_n < \frac{n}{2}.$$

**Lemma 2.3.** *Suppose  $s \geq 2$ ,  $a > 2e$  and  $[\frac{(2s-2t-1)n}{2b}, \frac{(s-t)n}{b}]$  contains an integer co-prime to  $n$  for some  $t \in [0, \lfloor \frac{s}{2} \rfloor - 1]$ . Then Proposition 2.1 holds.*

**Lemma 2.4.** *Suppose  $s \geq 2$ ,  $a > 2e$  and  $[\frac{(2s-2t-1)n}{2b}, \frac{(s-t)n}{b}]$  contains no integers co-prime to  $n$  for every  $t \in [0, \lfloor \frac{s}{2} \rfloor - 1]$ . Then the following results hold.*

- (i)  $\frac{n}{2b} < 3$  (where  $\frac{n}{2b}$  is the length of the interval  $[\frac{(2s-2t-1)n}{2b}, \frac{(s-t)n}{b}]$  for each  $t \in [0, \lfloor \frac{s}{2} \rfloor - 1]$ ).
- (ii) If  $s \geq 4$ , then  $[\frac{(2s-2t-1)n}{2b}, \frac{(s-t)n}{b}]$  contains exactly one integer for every  $t \in [0, \lfloor \frac{s}{2} \rfloor - 1]$ . Furthermore,  $\frac{n}{2b} < 2$ .
- (iii) Suppose that  $s \geq 4$ ,  $x \in [\frac{(2s-2t-1)n}{2b}, \frac{(s-t)n}{b}]$  and  $y \in [\frac{(2s-2t-3)n}{2b}, \frac{(s-t-1)n}{b}]$  for some  $t \in [0, \lfloor \frac{s}{2} \rfloor - 2]$ . Then  $\gcd(x, y, n) = 1$ .
- (iv) Suppose that  $s \geq 6$ ,  $x \in [\frac{(2s-2t-1)n}{2b}, \frac{(s-t)n}{b}]$  and  $z \in [\frac{(2s-2t-5)n}{2b}, \frac{(s-t-2)n}{b}]$  for some  $t \in [0, \lfloor \frac{s}{2} \rfloor - 3]$ . Then  $\gcd(x, z, n) > 1$  and  $5 \mid \gcd(x, z, n)$ . Furthermore,  $z = x - 5$  and  $\frac{n}{2b} < \frac{7}{5}$ .
- (v)  $s \leq 7$ .

Next we show that a further reduction of parameters can be done. Let

$$S = (eg) \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g) = (n_1g) \cdot (n_2g) \cdot (n_3g) \cdot (n_4g),$$

where  $e, a, b, c$  and  $g$  are as in Proposition 2.1 and  $n_1 = e, n_2 = c, n_3 = n - b$  and  $n_4 = n - a$ .

Let  $u$  be the greatest common divisor of  $n, n_1, n_2, n_3, n_4$ . If  $u > 1$ , we can consider  $G' = \langle ug \rangle$  and  $S = (\frac{n_1}{u}ug) \cdot (\frac{n_2}{u}ug) \cdot (\frac{n_3}{u}ug) \cdot (\frac{n_4}{u}ug)$ , where  $|G'| = \frac{n}{u}$  is less than  $n$ . Hence we can assume that  $u = 1$ . By the result of [11], we can assume that  $\gcd(n_i, n) > 1$  for  $i = 1, 2, 3, 4$ . Clearly, under this assumption, two of  $n_i$ 's have factor  $p$  and the other two have factor  $q$ .

We define  $i_0$  and  $j_0$  by

$$(2.1) \quad p^{i_0} = \min \left\{ \gcd(n_i, n) \mid p \mid n_i, i \in [1, 4] \right\} \quad q^{j_0} = \min \left\{ \gcd(n_i, n) \mid q \mid n_i, i \in [1, 4] \right\},$$

such that  $p^{i_0} < q^{j_0}$ .

**Proposition 2.5.** *It is sufficient to prove Proposition 2.1 under the following parameters:*

- (1)  $n \geq 75p^{i_0}$ ;
- (2)  $e \in \{p^{i_0}, q^{j_0}, 2q^{j_0}\}$  and  $a > 3e$ ;
- (3) If  $e \in \{q^{j_0}, 2q^{j_0}\}$ , then  $a \geq 6e$ ;
- (4)  $s \leq 7$ .

*Proof.* If  $i_0 = \alpha$  and  $j_0 = \beta$ , without loss of generality, let  $p|n_1, p|n_2$ , then the sum of  $p^\alpha|(n_1 + n_2)$  and  $q^\beta|(\nu n - n_3 - n_4) = (n_1 + n_2)$ , hence  $n|(n_1 + n_2)$ , which contradicts to that  $S$  is a minimal zero-sum sequence. Then we infer that  $\alpha + \beta > i_0 + j_0$  and  $\frac{n}{p^{i_0}} \geq 5q^{j_0} > 5p^{i_0}$ . If  $p^{i_0} \geq 15$ , then  $n \geq 75p^{i_0}$ . Otherwise, we have  $p^{i_0} \leq 13$  and  $\frac{n}{p^{i_0}} \geq \frac{1000}{13} > 75$ .

Now we renumber the sequence such that  $e < \frac{a}{3}$ . First we may assume that  $e = p^{i_0}$ . Then, for the purpose, we only need to consider the following three situations.

**The first situation:**  $2e > a$ , then  $a = q^{j_0}$ .

**Case 1.**  $a|b$ .

Let  $m = \frac{n+a}{a}, m_1 = \frac{n+2a}{a}, m_2 = \frac{n+3a}{a}, m_3 = \frac{n+4a}{a}$ .

If  $\gcd(n, m) = 1$  then

$$\begin{aligned} |me|_n &> \frac{n}{2}, \quad \text{since } \frac{n+a}{2} < \frac{n+a}{a}e \leq \frac{n+a}{a}(a-2) < \frac{5n}{7} + a - 1 < n, \\ |m(n-a)|_n &= n-a > \frac{n}{2}, |m(n-b)|_n = n-b > \frac{n}{2}. \end{aligned}$$

If  $\gcd(n, m) > 1$ , then  $j_0 = \beta$  and  $\gcd(n, m_1) = \gcd(n, m_2) = \gcd(n, m_3) = 1$ . Moreover,

$$|m_1e|_n > \frac{n}{2}, |m_2e|_n > \frac{n}{2}, |m_3e|_n > \frac{n}{2}, |m_1a|_n < \frac{n}{2}, |m_2a|_n < \frac{n}{2}, |m_3a|_n < \frac{n}{2}.$$

If  $b < \frac{n}{4}$ , we have  $|m_1(n-b)|_n = n-2b > \frac{n}{2}$ . If  $\frac{n}{4} < b < \frac{n}{3}$ , we have  $|m_3(n-b)|_n = 2n-4b > \frac{n}{2}$ . If  $\frac{n}{3} < b < \frac{n}{2}$ , we have  $|m_2(n-b)|_n = 2n-3b > \frac{n}{2}$ . Then we can find an integer  $m_i$  such that  $\gcd(n, m_i) = 1$  and all of  $|m_i e|_n, |m_i(n-b)|_n, |m_i(n-a)|_n$  are larger than  $\frac{n}{2}$ , which implies that  $\text{ind}(S) = 1$ .

**Case 2.**  $a|c$ .

Let  $m = \frac{n-a}{a}, m_1 = \frac{n-2a}{a}, m_2 = \frac{n+3a}{2a}, m_3 = \frac{n+5a}{2a}$ .

If  $\gcd(n, m) = 1$ , then  $\frac{n}{2} < |me|_n < n-10a$  and  $|mc|_n = n-c > \frac{n}{2}$ . For this case, if  $|m(n-b)|_n > \frac{n}{2}$ , we have done. Otherwise, it must hold  $a < |m(n-b)|_n$ . We get a renumbering:

$$(2.2) \quad e' = a, c' = |m(n-b)|_n, \{b', a'\} = \{c, n - |me|_n\},$$

and it is easy to check that  $a' \geq 6e'$ .

If  $\gcd(m, n) > 1$ , then  $a = q^\beta$ ,  $q|(p^\alpha - 1)$  and  $\gcd(n, m_1) = \gcd(n, m_2) = \gcd(n, m_3) = 1$ .

*Subcase 1.*  $c = 2ta$  for some integer  $t$ .

Let  $m = \frac{n+a}{2a}$ . Then  $|me|_n < \frac{n}{2}$ ,  $|mc|_n = \frac{c}{2} < \frac{n}{2}$ ,  $|m(n-a)|_n = \frac{n-a}{2} < \frac{n}{2}$ .

*Subcase 2.*  $c = (2t+1)a$  for some integer  $t$ .

If  $\frac{n}{4} > c$ , replace  $m$  by  $m_1$  and repeat the above process, we have  $|m_1(n-b)|_n > \frac{n}{2}$ ,  $|m_1c|_n > \frac{n}{2}$  and  $|m_1e|_n > \frac{n}{2}$ , which implies  $\text{ind}(S) = 1$ , or we can obtain a renumbering:

$$(2.3) \quad e' = 2a, c' = |m_1(n-b)|_n, \{b', a'\} = \{2c, n - |m_1e|_n\},$$

it also holds that  $a' \geq 6e'$ .

If  $\frac{n}{4} < c < \frac{n}{3}$ ,  $|m_3a|_n = \frac{n-5a}{2} < \frac{n}{2}$ . We have  $|m_3e|_n < \frac{n}{2}$  and  $|m_3c|_n = |\frac{n+5c}{2}|_n < \frac{n}{2}$ , exactly it belongs to  $(\frac{n}{8}, \frac{n}{3})$ . Then  $\text{ind}(S) = 1$ .

If  $\frac{n}{3} < c$ ,  $|m_2a|_n = \frac{n-3a}{2} < \frac{n}{2}$ . We have  $|m_2c|_n = |\frac{n+3c}{2}|_n < \frac{n}{4}$ ,  $|m_2e|_n < \frac{n}{2}$ , and hence  $\text{ind}(S) = 1$ .

**The second situation:**  $2e < a < 3e$  and  $e|b$ .

Let  $b = te$ , we have  $\frac{tn}{b} = \frac{n}{e}$ . Then  $\frac{tn}{b} - \frac{tn}{c} = \frac{t(a-e)n}{bc} > \frac{bn}{bc} > 2$ , and at least two integers  $m_1 = \frac{tn}{b} - 1 = \frac{n-e}{e}$ ,  $m_2 = m_1 - 1 = \frac{n-2e}{e}$  contained in  $(\frac{tn}{c}, \frac{tn}{b})$ .

It is easy to see that at least one of  $m_1, m_2$  is co-prime to  $n$ . Let  $m$  be one of them such that  $\text{gcd}(m, n) = 1$ . Then we have  $me < n$ ,  $mc \geq tn$ ,  $tn > mb \geq m_1b = t(n-2e) = tn - 2b > (t-1)n$  and  $2n < 2n - 4e + \frac{n}{e} - 2 = \frac{n-2e}{e}(2e+1) \leq ma < 3n$ . Hence

$$\begin{aligned} 3n &\geq |me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n \\ &\geq me + (mc - tn) + (tn - mb) + (3n - ma) = 3n. \end{aligned}$$

Where  $-4e + \frac{n}{e} - 2 > 0$  because  $n \geq \min\{\frac{pea}{2}, \frac{qea}{2}\} \geq \frac{5ea}{2} > 5e^2$  and  $\frac{n}{e} > 5e > 4e + 5$ . Thus  $\text{ind}(S) = 1$ .

**The third situation:**  $2e < a < 3e$  and  $e|c$ .

**Case 1.**  $a = q^{j_0}$  and  $b = (2t+1)a$ .

Let  $m = \frac{n-a}{a}$ . If  $\text{gcd}(n, m) = 1$ , then  $|me|_n < \frac{n}{2}$ ,  $|m(n-a)|_n = a < \frac{n}{2}$ ,  $|m(n-b)|_n = b < \frac{n}{2}$ . We have done.

If  $\text{gcd}(n, m) > 1$ , let  $m_1 = \frac{n+a}{2a}$ , then  $\text{gcd}(n, m_1) = 1$ .  $|m_1e|_n < \frac{n}{2}$ ,  $|m_1(n-a)|_n = \frac{n-a}{2} < \frac{n}{2}$ ,  $|m_1(n-b)|_n = \frac{n-b}{2} < \frac{n}{2}$ , hence  $\text{ind}(S) = 1$ .

**Case 2.**  $a = q^{j_0}$  and  $b = 2tq^{j_0}$ .

Let  $m = \frac{n-a}{a}$ . If  $\text{gcd}(n, m) = 1$ , then  $|me|_n < \frac{n}{2}$ ,  $|m(n-a)|_n = a < \frac{n}{2}$ ,  $|m(n-b)|_n = b < \frac{n}{2}$ . We have done.

If  $\text{gcd}(n, m) > 1$ , let  $m_1 = \frac{n-2a}{a}, m_2 = \frac{n+3a}{2a}, m_3 = \frac{n+a}{2a}$ .

If  $b < \frac{n}{4}$ , then  $|m_1e|_n < \frac{n}{2}$ ,  $|m_1(n-a)|_n = 2a < 2b < \frac{n}{2}$ ,  $|m_1(n-b)|_n = 2b < \frac{n}{2}$ . We have done.

If  $\frac{n}{4} < b < \frac{n}{3}$ , then  $|m_3e|_n < \frac{n}{2}$ ,  $|m_3(n-a)|_n = \frac{n-a}{2} < \frac{n}{2}$ ,  $|m_3(n-b)|_n = \frac{b}{2} < \frac{n}{2}$ . We have done.

If  $\frac{n}{3} < b < \frac{n}{2}$ , then  $|m_2e|_n < \frac{n}{2}$ ,  $|m_2(n-a)|_n = \frac{n-3a}{2} < \frac{n}{2}$ ,  $|m_2(n-b)|_n < \frac{n}{2}$ . We have done.

**Case 3.**  $a = 2q^{j_0}$  and  $b = 2tq^{j_0}$ .

Let  $m = \frac{n-q^{j_0}}{2q^{j_0}}$ . If  $\text{gcd}(n, m) = 1$ , then  $|me|_n < \frac{n}{2}$ ,  $|m(n-a)|_n = q^{j_0} < \frac{n}{2}$ ,  $|m(n-b)|_n = tq^{j_0} < \frac{n}{2}$ . We have done.

If  $\text{gcd}(n, m) > 1$ , let  $m_1 = \frac{3n-q^{j_0}}{2q^{j_0}}$ .

Then  $|m_1(n-a)|_n = \frac{a}{2} < \frac{n}{2}$ ,  $|m_1(n-b)|_n = \frac{b}{2} < \frac{n}{2}$ , and  $|m_1e|_n < m_1e - n < \frac{n}{2}$ , hence  $\text{ind}(S) = 1$ .

**Case 4.**  $a = 2q^{j_0}$  and  $b = (2t + 1)q^{j_0}$ .

Let  $m = \frac{n - q^{j_0}}{2q^{j_0}}$ . If  $\gcd(n, m) = 1$ , then  $|me|_n < \frac{n}{2}$ ,  $|m(n - a)|_n = q^{j_0} < \frac{n}{2}$ ,  $|m(n - b)|_n = n - tq^{j_0} > \frac{n}{2}$ . Clearly,  $t \geq 4$ .

We also have  $|mc|_n \in (\frac{c}{2}, \frac{n+c}{2})$ . If  $|mc|_n < \frac{n}{2}$ , then we have done. If  $|mc|_n > \frac{n}{2}$ , then  $n - |mc|_n > \frac{n-c}{2} \geq 10q^{j_0}$ , and we have renumbering

$$(2.4) \quad e' = q^{j_0}, c' = |me|_n, \{b', a'\} = \{|mb|_n, n - |mc|_n\}, \quad e' < a' \leq b' < c' < \frac{n}{2}.$$

Moreover, if  $p^{i_0} | (e' - a')$ , we have  $a' \geq 6e'$ . Then it always holds that  $a' \geq 6e'$  after this renumbering.

Up to now, we finish the renumbering. Hence, we can always assume that  $e \in \{p^{i_0}, q^{j_0}, 2q^{j_0}\}$  and  $a > 3e$ . Particularly,  $a \geq 6e$  when  $e \in \{q^{j_0}, 2q^{j_0}\}$ . Then in view of Lemmas 2.2, 2.3 and 2.4 and the above renumbering, from now on we may always assume that  $s \leq 7$ .  $\square$

Let  $k_1$  be the largest positive integer such that  $\lceil \frac{(k_1-1)n}{c} \rceil = \lceil \frac{(k_1-1)n}{b} \rceil$  and  $\frac{k_1 n}{c} \leq m < \frac{k_1 n}{b}$ . The existence of integer  $k_1$  has been proved in [11].

As mentioned above, we only need prove Proposition 2.1 under the parameters listed in Proposition 2.5. We now show that Proposition 2.1 holds through the following 3 propositions.

**Proposition 2.6.** *Suppose  $\lceil \frac{n}{c} \rceil < \lceil \frac{n}{b} \rceil$ , then Proposition 2.1 holds under the parameters listed in Proposition 2.5.*

**Proposition 2.7.** *Suppose  $\lceil \frac{n}{c} \rceil = \lceil \frac{n}{b} \rceil$ . Let  $k_1$  be the largest positive integer such that  $\lceil \frac{(k_1-1)n}{c} \rceil = \lceil \frac{(k_1-1)n}{b} \rceil$  and  $\frac{k_1 n}{c} \leq m_1 < \frac{k_1 n}{b}$  holds for some integer  $m_1$ . If  $k_1 > \frac{b}{a}$ , then Proposition 2.1 holds under the parameters listed in Proposition 2.5.*

**Proposition 2.8.** *Suppose  $\lceil \frac{n}{c} \rceil = \lceil \frac{n}{b} \rceil$ . Let  $k_1$  be the largest positive integer such that  $\lceil \frac{(k_1-1)n}{c} \rceil = \lceil \frac{(k_1-1)n}{b} \rceil$  and  $\frac{k_1 n}{c} \leq m_1 < \frac{k_1 n}{b}$  holds for some integer  $m_1$ . If  $k_1 \leq \frac{b}{a}$ , then Proposition 2.1 holds under the parameters listed in Proposition 2.5.*

### 3. PROOF OF PROPOSITION 2.6

In this section, we assume that  $\lceil \frac{n}{c} \rceil < \lceil \frac{n}{b} \rceil$ . Let  $m_1 = \lceil \frac{n}{c} \rceil$ . Then we have  $m_1 - 1 < \frac{n}{c} \leq m_1 < \frac{n}{b}$ . By Lemma 2.3 (1), it suffices to  $m$  and  $k$  such that  $\frac{kn}{c} \leq m < \frac{kn}{b}$ ,  $\gcd(m, n) = 1$ ,  $1 \leq k \leq b$ , and  $ma < n$ . So in what follows, we may always assume that  $\gcd(n, m_1) > 1$ .

**Lemma 3.1.** *Let  $e, a, b, c$  be parameters listed in Proposition 2.5. We have the following estimates:*

- (1) *If  $35|n$ , then  $n > 71e$ ;*
- (2) *If  $35|n$ , then  $n \geq 125e$  or  $a \geq 11e$ ;*
- (3) *If  $55|n$ , then  $n \geq 125e$ ;*
- (4) *If  $5|n$  and  $\gcd(77, n) = 1$ , then  $a \geq 125e$  for  $e = p^{i_0}$  and  $a \geq 25e$  for  $e \in \{q^{j_0}, 2q^{j_0}\}$ .*

This lemma can be showed simply and we omit the proof.

**Lemma 3.2.** *If  $[\frac{n}{c}, \frac{n}{b}]$  contains at least two integers, then  $\text{ind}(S) = 1$ .*

*Proof.* The proof is similar to that of Lemma 3.4 in [11].  $\square$

By Lemma 3.2, we may assume that  $\left[\frac{n}{c}, \frac{n}{b}\right]$  contains exactly one integer  $m_1$ , and thus

$$(3.1) \quad m_1 - 1 < \frac{n}{c} \leq m_1 < \frac{n}{b} < m_1 + 1.$$

Let  $l$  be the smallest integer such that  $\left[\frac{ln}{c}, \frac{ln}{b}\right]$  contains at least three integers. Clearly,  $l \geq 2$ . We claim that it holds either (referred to [11])

$$(3.2) \quad lm_1 - 2 < \frac{ln}{c} < \frac{ln}{b} < lm_1 + 3$$

or

$$(3.3) \quad lm_1 - 3 < \frac{ln}{c} < \frac{ln}{b} < lm_1 + 2.$$

**Lemma 3.3.** *Assume that*

$$5l - 2 < \frac{ln}{c} < 5l - 1 < 5l < 5l + 1 < \frac{ln}{b} < 5l + 2,$$

$$5(l - 1) - 1 < \frac{(l-1)n}{c} < 5(l - 1) < 5(l - 1) + 1 < \frac{(l-1)n}{b} < 5(l - 1) + 2,$$

and  $\gcd(5l - 1, n) = 1$ ,  $l \in [3, 9]$ ,  $5 \mid n$ . Then  $\text{ind}(S) = 1$ .

*Proof.* It is sufficient to show that  $ma < n$  for  $m = 5l - 1$ .

If  $e \leq \frac{a}{5}$ , then

$$ma = (5l - 1)(c - b + e) < \frac{5}{4}(5l - 1) \left( \frac{ln}{5l - 2} - \frac{ln}{5l + 2} \right) = \frac{(100l^2 - 20l)n}{100l^2 - 16} < n,$$

and we have done.

Next we can assume that  $e = p^{j_0} > \frac{a}{5}$ . It is easy to know that  $a \in \{q^{j_0}, 2q^{j_0}, 3q^{j_0}, 4q^{j_0}\}$ .

**Case 1.**  $5 \mid e$ .

If  $e = 5$ , then  $a \in \{17, 19, 21, 22, 23\}$ . When  $a \in \{17, 19, 23\}$ , we have  $\frac{n}{a} \geq 5q \geq 85 > (5l - 1)$  and we have done.

Moreover, we have  $\frac{n}{a} \geq \frac{1375}{22} > 62$  for  $a = 22$  and  $\frac{n}{a} \geq \frac{1225}{21} > 58$  for  $a = 21$ , both of them contradict to  $a > \frac{b}{8} > \frac{n}{48}$ .

If  $e \geq 125$ , we have  $n > 625e$ . Then

$$ma = (5l - 1)(c - b + e) < (5l - 1) \left( \frac{ln}{5l - 2} - \frac{ln}{5l + 2} + e \right) = \frac{(20l^2 - 4l)n}{25l^2 - 4} + (5l - 1)e < n,$$

we have done.

Let  $e = 25$ . If  $n \neq 125q^{j_0}$ , we have  $n \geq 25q^{j_0} \geq 25 \times 29 = 725$ . If  $q^{j_0} \geq 67$ , we have  $n \geq 635e$ . Both of these two situations imply that

$$ma < (5l - 1) \left( \frac{ln}{5l - 2} - \frac{ln}{5l + 2} + e \right) = \frac{(20l^2 - 4l)n}{25l^2 - 4} + (5l - 1)e < n.$$

Then we have done.

Let  $n = 125q^{j_0}$ . If  $a \leq 2q^{j_0}$  and  $n \geq \frac{125a}{2} > 62a$ , which contradicts to  $a > \frac{b}{8} > \frac{n}{48}$ .

If  $a = 3q^{j_0}$ , then  $e \mid c$ . Otherwise we have  $c \geq 28q^{j_0}$  and  $b = c + e - a \geq 25q^{j_0} + e > 8a$ , a contradiction. So  $c = 25(q^{j_0} - 1)$ , which implies  $\frac{n}{c} > 5$ , or  $c \geq 25(2q^{j_0} - 1)$ , which implies  $b \geq 47q^{j_0} > 8a$ , both of them give a contradiction.

We infer that  $a = 4q^{j_0}$ , hence  $q^{j_0} = q \in \{29, 31\}$ , similar to the above process, we obtain a contradiction.

**Case 2.**  $\gcd(5, e) = 1$ .

If  $e \geq 29$ , we have  $q^{j_0} \geq 125$  and  $n \geq 625e$ . Then it is easy to check that  $ma < n$ . We can assume that  $e = p \in \{7, 11, 13, 17, 19, 23\}$  and  $q^{j_0} = 25$ .

Moreover, we have  $c = p \times 24$  or  $b = 26 \times p$  (using the condition  $s \leq 7$ ), these imply  $\frac{n}{c} > 5$  or  $\frac{n}{b} < 5$ , a contradiction.  $\square$

**Lemma 3.4.** *If  $4 < \frac{n}{c} \leq 5 < \frac{n}{b} < 6$  and  $5|n$ , then  $\text{ind}(S) = 1$ .*

*Proof.* Since  $4 < \frac{n}{c} \leq 5 < \frac{n}{b} < 6$ ,  $n > 5b$ . Note that  $m_1 = \lceil \frac{n}{c} \rceil = 5$ .

If  $l = 2$ , since  $[\frac{ln}{c}, \frac{ln}{b})$  contains at least three integers, we must have  $8 < \frac{2n}{c} < 9 < 10 < 11 < \frac{2n}{b} < 12$ . Thus  $\frac{n}{6} < b < c < \frac{n}{4}$ . Let  $m = 9$  and  $k = 2$ . Then by Proposition 2.5,  $9a = 9 \times (c - b + e) < 9 \times (\frac{n}{4} - \frac{n}{6} + e) = \frac{3n}{4} + 9e < n$  or  $9a \leq \frac{6}{5} \times 9 \times (c - b) < \frac{54}{5} \times (\frac{n}{4} - \frac{n}{6}) = \frac{9n}{10} < n$ , and we are done.

Next assume that  $l \geq 3$ . Since  $[\frac{ln}{c}, \frac{ln}{b})$  contains at least three integers and  $5l - 3 < \frac{ln}{c} < \frac{ln}{b} \leq 5l + 3$ , we can divide the proof into three cases.

**Case 1.**  $5l + 2 < \frac{ln}{b} \leq 5l + 3$ . Then  $\frac{2}{l} \leq \frac{n}{b} - 5 \leq \frac{3}{l}$ .

For  $\gamma \in [\frac{l+1}{2}, l - 1]$ , since  $\gamma(\frac{n}{b} - 5) > \frac{l}{2} \cdot \frac{2}{l} = 1$  and thus  $\frac{\gamma n}{c} \leq 5\gamma < 5\gamma + 1 < \frac{\gamma n}{b}$ . By the minimality of  $l$  we infer that

$$(3.4) \quad 5\gamma - 1 < \frac{\gamma n}{c} \leq 5\gamma < 5\gamma + 1 < \frac{\gamma n}{b} < 5\gamma + 2.$$

Let  $\gamma = l - 1$ . We have  $(5(l - 1) - 1)(b + a - e) = (5(l - 1) - 1)c < (l - 1)n < (5(l - 1) + 2)b$  and thus  $(5l - 6)(a - e) < 3b$ .

If  $l \geq 16$ , let  $k = l$  and let  $m$  be an integer in  $[\frac{ln}{c}, \frac{ln}{b})$  which is co-prime to  $n$ . Then  $m \leq 5l + 2$  and

$$ma \leq (5l + 2) < \frac{5l + 2}{5l - 6} \times \frac{3}{2} \times (5l - 6)(a - e) < \frac{5 \times 16 + 2}{5 \times 16 - 6} \times \frac{3}{2} \times 3b < 5b \leq n,$$

and we have done.

Next assume that  $l \in [6, 15]$ .

If  $\gcd(5l - 4, n) = 1$ , let  $m = 5l - 4$  and  $k = l - 1$ . Then by (3.4)  $\frac{kn}{c} \leq m < \frac{kn}{b}$  and

$$ma = (5l - 4) < \frac{5l - 4}{5l - 6} \times \frac{3}{2} \times (5l - 6)(a - e) < \frac{5 \times 6 - 4}{5 \times 6 - 6} \times \frac{3}{2} \times 3b < 5b \leq n,$$

as desired. Thus we may assume that  $\gcd(5l - 4, n) > 1$ .

Applying (3.4) with  $\gamma = l - 2$ , we have  $\gcd(5l - 9, n) = 1$  and  $5l - 11 < \frac{(l-2)n}{c} \leq 5l - 10 < 5l - 9 < \frac{(l-2)n}{b} \leq 5l - 8$ . Thus  $\frac{(l-2)n}{5l-8} \leq b < c < \frac{(l-2)n}{5l-11}$ . Let  $m = 5l - 9$  and  $k = l - 2$ , we have

$$ma = (5l - 9)a < \frac{3}{2} \times (5l - 9) \times \left( \frac{(l-2)n}{5l-11} - \frac{(l-2)n}{5l-8} \right) < n,$$

and we have done.

Finally, assume that  $l \leq 5$ .

If  $l \in [4, 5]$ , applying (3.4) with  $\gamma = 3$ , we have  $14 < \frac{3n}{c} \leq 15 < 16 < \frac{3n}{b} \leq 17$ . then  $\frac{3n}{17} \leq b < c < \frac{3n}{14}$ . Note that  $\gcd(n, 16) = 1$ . Let  $m = 16$  and  $k = 3$ . Then

$$ma = 16a < 16 \times \frac{3}{2} \times \left( \frac{3n}{14} - \frac{3n}{17} \right) = \frac{27 \times 16n}{28 \times 17} < n,$$



and we have done.

If  $l = 3$ , we have  $\frac{3n}{c} \leq 15 < 16 < 17 < \frac{3n}{b} \leq 18$ . If  $\frac{3n}{c} > 14$ , then  $c < \frac{3n}{14}$ . Let  $k = 3$  and  $m = 16$ . By Lemma 3.1, we have  $16a < 16 \times \frac{11}{10} \times \left(\frac{3n}{14} - \frac{n}{6}\right) = \frac{88n}{105} < n$ , or  $16a < 16 \times \left(\frac{3n}{14} - \frac{n}{6} + \frac{n}{125}\right) < n$ , as desired. If  $\frac{3n}{c} \leq 14$ , we have  $13 < \frac{3n}{c} \leq 14$ . Applying (3.4) with  $\gamma = 2$ , we have  $9 < \frac{2n}{c} \leq 10 < 11 < \frac{2n}{b} \leq 12$ , and then  $\frac{n}{6} \leq b < c < \frac{2n}{9}$ . Note that either  $\gcd(11, n) = 1$  or  $\gcd(n, 14) = 1$ . Now let  $m = 11$  and  $k = 2$  if  $\gcd(n, 11) = 1$ , or let  $m = 14$  and  $k = 3$  if  $\gcd(n, 14) = 1$ . Then

$$ma \leq 14a < 14 \times \frac{3}{2} \times \left(\frac{3n}{13} - \frac{1n}{6}\right) = \frac{77n}{78} < n,$$

and we have done.

This completes the proof of Case 1.

**Case 2.**  $\frac{ln}{b} \leq 5l + 2$  and  $5l - 3 < \frac{ln}{c} \leq 5l - 2$ . This case can be proved in a similar way to Case 1.

**Case 3.**  $\frac{ln}{b} \leq 5l + 2$  and  $\frac{ln}{c} > 5l - 2$ . Thus  $5l - 2 < \frac{ln}{c} \leq 5l - 1 < 5l < 5l + 1 < \frac{ln}{b} \leq 5l + 2$ . This implies that every integer in  $\left[\frac{ln}{c}, \frac{ln}{b}\right)$  is less than  $5l + 2$ . By the minimality of  $l$ , we must have one of the following holds.

- (i)  $5l - 6 < \frac{(l-1)n}{c} \leq 5l - 5 < \frac{(l-1)n}{b} \leq 5l - 4$ .
- (ii)  $5l - 6 < \frac{(l-1)n}{c} \leq 5l - 5 < 5l - 4 < \frac{(l-1)n}{b} \leq 5l - 3$ .
- (iii)  $5l - 7 < \frac{(l-1)n}{c} \leq 5l - 6 < 5l - 5 < \frac{(l-1)n}{b} \leq 5l - 4$ .

We divide the proof into three subcases according to the above three situations.

*Subcase 3.1.* (i) holds. Let  $k = l$  and  $m$  be an integer in  $\left[\frac{ln}{c}, \frac{ln}{b}\right)$  which is co-prime to  $n$ . Note that  $m \leq 5l + 1$ , then

$$ma \leq (5l + 1)a < \frac{3}{2} \times (5l + 1) \times \left(\frac{(l-1)n}{5l-6} - \frac{(l-1)n}{5l-4}\right) = \frac{3(l-1)(5l+1)n}{(5l-6)(5l-4)} < n,$$

and we have done.

*Subcase 3.2.* (ii) holds.

If  $l \geq 10$ , then let  $k = l$  and  $m$  be an integer in  $\left[\frac{ln}{c}, \frac{ln}{b}\right)$  which is co-prime to  $n$ . Note that  $m \leq 5l + 1$ , then

$$ma \leq (5l + 1)a < \frac{3}{2} \times (5l + 1) \times \left(\frac{(l-1)n}{5l-6} - \frac{(l-1)n}{5l-3}\right) = \frac{9(l-1)(5l+1)n}{2(5l-6)(5l-3)} < n,$$

and we have done.

Next assume that  $l \in [3, 9]$ . If  $\gcd(5l - 4, n) = 1$ , let  $m = 5l - 4$  and  $k = l - 1$ . Then

$$ma \leq (5l - 4)a < \frac{3}{2} \times (5l - 4) \times \left(\frac{(l-1)n}{5l-6} - \frac{(l-1)n}{5l-3}\right) = \frac{9(l-1)(5l-4)n}{2(5l-6)(5l-3)} \leq n,$$

as desired. Hence we may assume that  $\gcd(5l - 4, n) > 1$ . This implies that  $\gcd(5l - 1, n) = 1$ . Now let  $m = 5l - 1$  and  $k = l$ , by Lemma 3.3,  $\text{ind}(S) = 1$ .

*Subcase 3.3.* (iii) holds. This subcase can be proved in a similar way to Subcase 3.2.  $\square$

**Lemma 3.5.** *If  $6 < \frac{n}{c} \leq 7 < \frac{n}{b} < 8$  and  $7|n$ , then  $\text{ind}(S) = 1$ .*

*Proof.* Since  $6 < \frac{n}{c} \leq 7 < \frac{n}{b} < 8$ , we have  $\frac{n}{8} < b < \frac{n}{7} \leq c < \frac{n}{6}$ . Note that  $m_1 = 7$ .

If  $l = 2$ , then  $12 < \frac{2n}{c} \leq 13 < 14 < 15 < \frac{2n}{b} < 16$ . If  $\gcd(15, n) = 1$ , let  $m = 15$  and  $k = 2$ , otherwise let  $m = 13$  and  $k = 2$ . Then

$$ma \leq 15a \leq 15 \times 32(c - b) < \frac{45}{2} \times \left( \frac{n}{6} - \frac{n}{8} \right) < n,$$

and we have done.

Next assume that  $l \geq 3$ . Recall that  $7l - 3 < \frac{ln}{c} \leq 7l < \frac{ln}{b} < 7l + 3$ . We distinguish two cases according to the number of integers contained in  $[\frac{ln}{c}, \frac{ln}{b})$ .

**Case 1.** There exist exactly three integers in  $[\frac{ln}{c}, \frac{ln}{b})$ .

Then  $7l - t < \frac{ln}{c} \leq 7l - t + 1 < 7l - t + 2 < 7l - t + 3 < \frac{ln}{b} \leq 7l - t + 4$  for some  $t \in [1, 3]$ . Let  $k = l$  and  $m \in [7l - t + 1, 7l - t + 3]$  such that  $\gcd(n, m) = 1$ . Then

$$\begin{aligned} ma &\leq (7l - t + 3)a \leq \frac{3(7l - t + 2)}{2}(c - b) \\ &< \frac{3(7l - t + 3)}{2} \left( \frac{ln}{7l - t} - \frac{ln}{7l - t + 4} \right) = \frac{(7l - t + 3) \times 6ln}{(7l - t)(7l - t + 4)} < n, \end{aligned}$$

and we have done.

**Case 2.** There exist exactly four integers in  $[\frac{ln}{c}, \frac{ln}{b})$ .

First we have  $7l - 2 < \frac{ln}{c} \leq 7l - 1 < 7l < 7l + 1 < 7l + 2 < \frac{ln}{b} \leq 7l + 3$  or  $7l - 3 < \frac{ln}{c} \leq 7l - 2 < 7l - 1 < 7l < 7l + 1 < \frac{ln}{b} \leq 7l + 2$ . Then there exists  $m \leq 7l + 1$  contained in  $[\frac{ln}{c}, \frac{ln}{b})$  such that  $\gcd(n, m) = 1$ .

By the minimality of  $l$ , we have

$$7(l - 1) - 1 < \frac{(l - 1)n}{c} \leq 7(l - 1) < 7(l - 1) + 1 < \frac{(l - 1)n}{b} \leq 7(l - 1) + 2,$$

or

$$7(l - 1) - 2 < \frac{(l - 1)n}{c} \leq 7(l - 1) - 1 < 7(l - 1) < \frac{(l - 1)n}{b} \leq 7(l - 1) + 1.$$

Then

$$ma \leq (7l - 1)a < \frac{3(7l - 1)}{2} \times \left( \frac{(l - 1)n}{7l - 8} - \frac{(l - 1)n}{7l - 5} \right) < n,$$

or

$$ma \leq (7l - 1)a < \frac{3(7l - 1)}{2} \times \left( \frac{(l - 1)n}{7l - 9} - \frac{(l - 1)n}{7l - 6} \right) \leq n,$$

and we have done.  $\square$

Now we are in a position to prove Proposition 2.6.

*Proof of Proposition 2.6.*

Recall that either  $m_1 = 5$  or  $m_1 = 7$  or  $m_1 \geq 10$ . By Lemmas 3.5 and 3.6 we may assume  $m_1 \geq 10$ . Then  $n \geq m_1 b \geq 10b$ . Let  $k = l$  and let  $m$  be one of the integers in  $[\frac{ln}{c}, \frac{ln}{b})$  which is co-prime to  $n$ . Recall that we have either (3.3) holds or (3.4) holds.

If (3.2) holds, then  $(lm_1 - 2)(b + a - e) = (lm_1 - 2)c < ln \leq (lm_1 + 3)b$ , so  $(lm_1 - 2)(a - e) < 5b$ . Note that  $m \leq lm_1 + 2$  and  $l \geq 2$ , then

$$ma \leq (lm_1 + 2)a = \frac{lm_1 + 2}{lm_1 - 2} \times \frac{a}{a - e} \times (lm_1 - 2)(a - e) < \frac{2 \times 10 + 2}{2 \times 10 - 2} \times \frac{3}{2} \times 5b < 10b \leq n,$$

and we are done.

If (3.3) holds, then  $(lm_1 - 3)(b + a - e) = (lm_1 - 3)c < ln \leq (lm_1 + 2)b$ , so  $(lm_1 - 3)(a - e) < 5b$ . Note that  $m \leq lm_1 + 1$  and  $l \geq 2$ , then

$$ma \leq (lm_1 + 1)a = \frac{lm_1 + 1}{lm_1 - 3} \times \frac{a}{a - e} \times (lm_1 - 3)(a - e) < \frac{2 \times 10 + 1}{2 \times 10 - 3} \times \frac{3}{2} \times 5b < 10b \leq n,$$

and we are done.

#### 4. PROOF OF PROPOSITION 2.7

In this section, we always assume that  $\lceil \frac{n}{c} \rceil = \lceil \frac{n}{b} \rceil$ , so  $k_1 \geq 2$ , and we also assume that  $k_1 > \frac{b}{a}$ . Proposition 2.7 can be proved through the following three lemmas.

**Lemma 4.1.** *If the assumption is as in Proposition 2.7, then  $k_1 < 4$ .*

*Proof.* If  $k_1 \geq 4$ , then  $\frac{(k_1 - 1)n}{b} - \frac{(k_1 - 1)n}{c} = \frac{(a - e)(k_1 - 1)n}{bc} \geq \frac{2a}{3} \frac{3k_1 n}{4bc} > 1$ , a contradiction.  $\square$

**Lemma 4.2.** *If the assumption is as in Proposition 2.7, then  $k_1 \neq 3$ .*

*Proof.* If  $a \geq 4e$ , then  $\frac{(k_1 - 1)n}{b} - \frac{(k_1 - 1)n}{c} = \frac{(a - e)2n}{bc} \geq \frac{3a}{4} \frac{2n}{bc} > 1$ , a contradiction. Hence we assume that  $3e < a < 4e$ , and  $e < \frac{3p^{i_0}}{2}$ .

If  $\frac{n}{c} > \frac{9}{4}$ , then  $\frac{(k_1 - 1)n}{b} - \frac{(k_1 - 1)n}{c} = \frac{(a - e)2n}{bc} \geq \frac{2a}{3} \frac{2n}{bc} > 1$ , a contradiction.

If  $\frac{n}{c} < \frac{9}{4} < \frac{n}{b} < \frac{5}{2}$ , then  $9a > 3b > \frac{6n}{5}$ , and  $n < \frac{45a}{6} < \frac{45 \times 4 \times \frac{3}{2} p^{i_0}}{6} = 45p^{i_0}$ , a contradiction.

If  $\frac{n}{c} < \frac{n}{b} < \frac{9}{4}$ , then  $9a > 3b > \frac{4n}{3}$ ,  $n < 27e < \frac{81}{2} p^{i_0}$ , a contradiction.  $\square$

**Lemma 4.3.** *If the assumption is as in Proposition 2.7 and  $k_1 = 2$ , then  $\text{ind}(S) = 1$ .*

*Proof.* If  $\frac{n}{c} > 3$ , then  $\frac{n}{b} - \frac{n}{c} = \frac{(a - e)n}{bc} \geq \frac{2a}{3} \frac{n}{bc} > 1$ , a contradiction.

If  $\frac{n}{c} \leq 3 < \frac{n}{b}$ , we have  $n < 3c < 2n$ ,  $3a < 3b < n$ . Let  $m = 3$ , then  $\gcd(n, m) = 1$  and  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n = me + (mc - n) + (n - mb) + (n - ma) = n$ , we have done.

If  $\frac{n}{c} < \frac{n}{b} < 3$ , then  $\frac{n}{3} < b < 2a$ , and  $2n < 6c < 3n$ ,  $2n < 6b < 3n$ ,  $6a > 3b > n$ .  $6e < 2a < n$ . Let  $m = 6$ , then  $\gcd(n, m) = 1$ , and  $3n \geq |me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n \geq me + (mc - 2n) + (3n - mb) + (2n - ma) = 3n$ , we have done.  $\square$

#### 5. PROOF OF PROPOSITION 2.8

In this section, we always assume that  $\lceil \frac{n}{c} \rceil = \lceil \frac{n}{b} \rceil$ , so  $k_1 \geq 2$ , and we also assume that  $k_1 < \frac{b}{a}$ , hence  $s \geq k_1$ .

**Lemma 5.1.** *If the assumption is as in Proposition 2.8, then  $k_1 \neq 7$ .*

*Proof.* If  $k_1 = 7$ , then  $s = 7$ , and  $\frac{(k_1 - 1)n}{b} - \frac{(k_1 - 1)n}{c} = \frac{(a - e)6n}{bc} \geq \frac{2 \times 8a}{3b} \frac{3}{4} \frac{n}{c} > 1$ , a contradiction.  $\square$

**Lemma 5.2.** *If the assumption is as in Proposition 2.8 and  $k_1 = 6$ , then  $\text{ind}(S) = 1$ .*

*Proof.* If  $k_1 = 6$ , we have  $\frac{n}{c} < \frac{12}{5}$ , otherwise  $\frac{(k_1 - 1)n}{b} - \frac{(k_1 - 1)n}{c} = \frac{(a - e)5n}{bc} \geq \frac{2 \times 8a}{3b} \frac{5}{8} \frac{n}{c} > \frac{10n}{24c} \geq 1$ , a contradiction. So we have  $10 < \frac{5n}{c} < \frac{5n}{b} \leq 11$  or  $11 < \frac{5n}{c} < \frac{5n}{b} \leq 12$ .

**Case 1.**  $10 < \frac{5n}{c} < \frac{5n}{b} \leq 11$ .

It holds that  $12 < \frac{6n}{c} \leq 13 < \frac{6n}{b} \leq \frac{66}{5}$  and  $16 < \frac{8n}{c} \leq 17 < \frac{8n}{b} \leq \frac{88}{5}$ .

If  $17a \geq n$ , then  $8n < 18b < 18c < 9n$  and  $18e < 6a < b < n$ . Let  $m = 18$ , then  $\gcd(n, m) = 1$  and  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n \geq 18e + (18c - 8n) + (9n - 18b) + (2n - 18a) = 3n$ , hence  $\text{ind}(S) = 1$ .

Assume that  $17a < n$ , then at least one of  $\{13, 17\}$  co-prime to  $n$  through Lemma 2.4(iv), which says  $5|n$ . Then we have done.

**Case 2.**  $11 < \frac{5n}{c} < \frac{5n}{b} \leq 12$ .

It holds that  $\frac{77}{5} < \frac{7n}{c} < 16 < \frac{7n}{b} \leq \frac{84}{5}$ . Since  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times (\frac{5n}{11} - \frac{5n}{12}) = \frac{5n}{88} < \frac{n}{17}$ , we have  $16a < 17a < n$ . Let  $m = 16$ , then  $\gcd(n, m) = 1$  and  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n \geq 16e + (16c - 7n) + (7n - 16b) + (n - 16a) = n$ , hence  $\text{ind}(S) = 1$ .  $\square$

**Lemma 5.3.** *If the assumption is as in Proposition 2.8 and  $k_1 = 5$ , then  $\text{ind}(S) = 1$ .*

*Proof.* If  $k_1 = 5$ , we have  $\frac{n}{c} < 3$ , otherwise  $\frac{(k_1-1)n}{b} - \frac{(k_1-1)n}{c} = \frac{(a-e)4n}{bc} \geq \frac{2 \times 4a}{3b} \frac{n}{c} > \frac{n}{c} \geq 1$ , a contradiction. So it holds  $8 + t < \frac{4n}{c} < \frac{4n}{b} \leq 9 + t$  for some  $t = 0, 1, 2, 3$ .

**Case 1.**  $t = 0$ . We have  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times (\frac{n}{2} - \frac{4n}{9}) = \frac{n}{12}$ .

If  $\gcd(n, 11) = 1$ , we have  $10 < \frac{5n}{c} \leq 11 < \frac{5n}{b} < \frac{45}{4}$ . Let  $m = 11$ , then  $\text{ind}(S) = 1$ .

If  $15a > n$ , we have  $14 < \frac{7n}{c} \leq 15 < \frac{7n}{b} < \frac{63}{4} < 16$  and  $7n < 16b < 16c < 8n$  and  $16e < 6a < n$ . Let  $m = 16$ , then  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n \geq 16e + (16c - 7n) + (8n - 16b) + (2n - 16a) = 3n$ , hence  $\text{ind}(S) = 1$ .

If  $15a < n$  and  $\gcd(n, 5) = 1$ , let  $m = 15$ , we have  $\text{ind}(S) = 1$ .

If  $15a \leq n$  and  $5|n, 11|n$ , we have  $12 < \frac{6n}{c} \leq 13 < \frac{6n}{b} < \frac{27}{2}$ . Let  $m = 13$ , we have  $\text{ind}(S) = 1$ .

**Case 2.**  $t = 1$ . We have  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times (\frac{4n}{9} - \frac{4n}{10}) = \frac{n}{15}$ . Since  $\frac{45}{4} < \frac{5n}{c} < 12 < \frac{5n}{b} < \frac{50}{4}$ , let  $m = 12$ , then  $\text{ind}(S) = 1$ .

**Case 3.**  $t = 2$ . We have  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times (\frac{4n}{10} - \frac{4n}{11}) = \frac{3n}{55} < \frac{n}{18}$ . Since  $15 = \frac{60}{4} < \frac{6n}{c} < 16 < \frac{6n}{b} < \frac{66}{4} < 17$ , let  $m = 16$ , then  $\text{ind}(S) = 1$ .

**Case 4.**  $t = 3$ . We have  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times (\frac{4n}{11} - \frac{4n}{12}) = \frac{n}{22}$ . We have

$$\begin{aligned} \frac{55}{4} &< \frac{5n}{c} < 14 < \frac{5n}{b} < 15, \\ \frac{66}{4} &< \frac{6n}{c} < 17 < \frac{6n}{b} < 18, \\ \frac{77}{4} &< \frac{7n}{c} < 20 < \frac{7n}{b} < 21, \end{aligned}$$

. At least one of  $\{14, 17, 20\}$  coprime to  $n$ . Let  $m$  be one of  $\{14, 17, 20\}$  such that  $\gcd(n, m) = 1$ , then  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n = n$  and  $\text{ind}(S) = 1$ .  $\square$

**Lemma 5.4.** *If the assumption is as in Proposition 2.8 and  $k_1 = 4$ , then  $\text{ind}(S) = 1$ .*

*Proof.* If  $k_1 = 4$ , we have  $s \geq 4$  and  $\frac{n}{b} < 4$ . So it holds  $6 + t < \frac{3n}{c} < \frac{3n}{b} \leq 7 + t$  for some  $t = 0, 1, 2, 3, 4, 5$ .

**Case 1.**  $t = 0$ . We have  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times (\frac{n}{2} - \frac{3n}{7}) = \frac{3n}{28} < \frac{n}{9}$ , and  $8 < \frac{4n}{c} < 9 < \frac{4n}{b} < \frac{28}{3}$ . Let  $m = 9$ , then  $\text{ind}(S) = 1$ .

**Case 2.**  $t = 1$ . We have  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{3n}{7} - \frac{3n}{8}\right) = \frac{9n}{112} < \frac{n}{12}$ . If  $\frac{35}{3} < \frac{5n}{c} < 12 < \frac{5n}{b} < \frac{40}{3}$ , let  $m = 12$ , then  $\text{ind}(S) = 1$ . If  $12 < \frac{5n}{c} \leq 13 < \frac{5n}{b} < \frac{40}{3}$ , we have  $a < \frac{3}{2} \times \left(\frac{5n}{12} - \frac{3n}{8}\right) = \frac{n}{16}$ , hence  $\text{ind}(S) = 1$  in case of  $\gcd(n, 13) = 1$ . We also have  $\text{ind}(S) = 1$  in case of  $\gcd(n, 13) = 1$  since  $\frac{28}{3} < \frac{4n}{c} < 10 < \frac{4n}{b} < \frac{32}{3}$ .

Assume that  $5|n, 13|n$  and  $12 < \frac{5n}{c} \leq 13 < \frac{5n}{b} < \frac{40}{3}$ . Hence we have  $\frac{84}{5} < \frac{7n}{c} < \frac{7n}{b} < \frac{56}{3}$ .

If  $18a > n$ , let  $m = 19$ . Then  $me = 19 < n$ ,  $7n < mb < mc < \frac{96c}{5} = \frac{8}{7} \times \frac{84c}{5} < 8n$ . Hence we have  $\gcd(n, m) = 1$  and  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n \geq 19e + (19c - 7n) + (8n - 19b) + (2n - 19a) = 3n$ . So  $\text{ind}(S) = 1$ .

If  $18a < n$ , there exists  $m \in \{17, 18\}$  such that  $\frac{7n}{c} \leq m < \frac{7n}{b}$ ,  $ma < n$  and  $\gcd(n, m) = 1$ , then we have done.

**Case 3.**  $t = 2$ . We have  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{3n}{8} - \frac{3n}{9}\right) = \frac{n}{16}$ .

If  $\gcd(n, 11) = 1$  or  $\gcd(n, 7) = 1$ , by inequalities  $\frac{32}{3} < \frac{4n}{c} \leq 11 < \frac{4n}{b} < 12, \frac{40}{3} < \frac{5n}{c} \leq 14 < \frac{5n}{b} < 15$ , it is easy to show that  $\text{ind}(S) = 1$ .

Assume that  $11|n, 7|n$ . We have  $16 < \frac{6n}{c} \leq 17 < \frac{6n}{b} < 18$ .

If  $17a < n$ , let  $m = 17$ , we have done.

If  $17a > n$ , let  $m = 18$ . Then  $6n < mb < mc = \frac{9}{8}16c < \frac{7}{6}16c < 7n$ , and

$$|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n \geq 18e + (18c - 6n) + (7n - 18b) + (2n - 18a) = 3n.$$

So  $\text{ind}(S) = 1$ .

**Case 4.**  $t = 3$ . We have  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{3n}{9} - \frac{3n}{10}\right) = \frac{n}{20}$ , and  $15 < \frac{5n}{c} < 16 < \frac{5n}{b} < \frac{50}{3} < 17$ . Let  $m = 16$ , then  $\text{ind}(S) = 1$ .

**Case 5.**  $t = 4$ . We have  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{3n}{10} - \frac{3n}{11}\right) < \frac{n}{24}$  and  $\frac{50}{3} < \frac{5n}{c} < 16 < \frac{5n}{b} < \frac{55}{3}$ .

If  $\frac{50}{3} < \frac{5n}{c} < 18 < \frac{5n}{b} < \frac{55}{3}$ , let  $m = 18$ . Then  $\text{ind}(S) = 1$ .

If  $\frac{50}{3} < \frac{5n}{c} \leq 17 < \frac{5n}{b} < 18$ , we have  $30a < n$ . Then  $n > 30a > \frac{15b}{4} > \frac{15}{4} \times \frac{5n}{18} = \frac{25n}{24} > n$ , it is a contradiction.

**Case 6.**  $t = 5$ . We have  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{3n}{11} - \frac{3n}{12}\right) < \frac{n}{29}$  and  $\frac{11}{3} < \frac{n}{c} < \frac{n}{b} < 4$ . So

$$\frac{44}{3} < \frac{4n}{c} \leq 15 < \frac{4n}{b} < 16,$$

$$\frac{55}{3} < \frac{5n}{c} \leq 19 < \frac{5n}{b} < 20,$$

$$22 < \frac{n}{c} \leq 23 < \frac{n}{b} < 24.$$

Then there exists at least one of integers 15, 19, 23 coprime to  $n$ . So it is clear that  $\text{ind}(S) = 1$ .  $\square$

**Lemma 5.5.** *If the assumption is as in Proposition 2.8 and  $k_1 = 3$ , then  $\text{ind}(S) = 1$ .*

*Proof.* If  $k_1 = 3$ , we have  $\frac{n}{b} < 6$ . So it holds  $4 + t < \frac{2n}{c} < \frac{2n}{b} \leq 5 + t$  for some integer  $t \in [0, 7]$ .

**Case 1.**  $t = 0$ .  $6 < \frac{3n}{c} \leq 7 < \frac{3n}{b} \leq \frac{15}{2}$ .

If  $8a > n$ , let  $m = 8$ . Then  $3n < 8b < 8c < 4n$ ,  $8e < 3a < b < n$  and  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n \geq 8e + (8c - 3n) + (4n - 8b) + (2n - 8a) = 3n$ . So  $\text{ind}(S) = 1$ .

If  $8a < n$ , since  $8 < \frac{4n}{c} < 9 < \frac{4n}{b} \leq 10$ , let  $m = 9$ . Then  $\text{ind}(S) = 1$ .

**Case 2.**  $t = 1$ . We have  $\frac{15}{2} < \frac{3n}{c} < 8 < \frac{3n}{b} < 9$  and  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{2n}{5} - \frac{n}{3}\right) = \frac{n}{10}$ . Let  $m = 8$ , then  $\gcd(n, m) = 1$  and  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n = n$ , hence  $\text{ind}(S) = 1$ .

**Case 3.**  $t = 2$ . We have  $9 < \frac{3n}{c} < 10 < \frac{3n}{b} < \frac{21}{2}$  and  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{n}{3} - \frac{2n}{7}\right) = \frac{n}{14}$ .

If  $17a \geq n$ , let  $m = 18$ , then  $5n < 18b < 18c = \frac{6}{5} \times 15c < 6n$  and  $18e < 6a < n$ , we have  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n \geq 3n$ , hence  $\text{ind}(S) = 1$ .

If  $17a < n$  and  $15 < \frac{5n}{c} < 16 < \frac{5n}{b} \leq \frac{35}{2}$ , let  $m = 16$ . Then  $\text{ind}(S) = 1$ .

Assume that  $16 < \frac{5n}{c} \leq 17 < \frac{5n}{b} \leq \frac{35}{2}$ , then  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{n}{24}$ . We also have  $9 < \frac{3n}{c} \leq 10 < \frac{3n}{b} \leq \frac{21}{2}$  and  $12 < \frac{4n}{c} < 13 < \frac{4n}{b} < 14$ . Then at least one of integers 10, 13, 17 is co-prime to  $n$ , and we have done.

**Case 4.**  $t = 3$ . We have  $\frac{7}{2} < \frac{n}{c} < \frac{n}{b} < 4$  and  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{2n}{7} - \frac{n}{4}\right) < \frac{n}{18}$ .

At first we have  $\frac{35}{2} < \frac{5n}{c} < \frac{5n}{b} < 20$ .

If  $\frac{5n}{c} < 18 < \frac{5n}{b}$ , let  $m = 18$ , then we have done.

If  $18 < \frac{5n}{c} \leq 19 < \frac{5n}{b} < 20$ , we have  $a < \frac{n}{24}$ . Since  $\frac{21}{2} < \frac{3n}{c} \leq 11 < \frac{3n}{b} < 12$ ,  $14 < \frac{4n}{c} < 15 < \frac{4n}{b} < 16$  and at least one of integers 11, 15, 19 is co-prime to  $n$ , then it is easy to show that  $\text{ind}(S) = 1$ .

**Case 5.**  $t = 4$ . We have  $4 < \frac{n}{c} < \frac{n}{b} < \frac{9}{2}$  and  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{n}{4} - \frac{2n}{9}\right) = \frac{n}{24}$ .

We also have

$$\begin{aligned} 12 &< \frac{3n}{c} \leq 13 < \frac{3n}{b} < \frac{27}{2}, \\ 16 &< \frac{4n}{c} \leq 17 < \frac{4n}{b} < 18, \\ 20 &< \frac{5n}{c} \leq m_1 < \frac{5n}{b} < \frac{45}{2}, \end{aligned}$$

where  $m \in \{21, 22\}$ . It is easy to see that at least one of integers 13, 17,  $m_1$  is co-prime to  $n$ . Then  $\text{ind}(S) = 1$ .

**Case 6.**  $t = 5$ . We have  $\frac{9}{2} < \frac{n}{c} < \frac{n}{b} \leq 5$  and  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{2n}{9} - \frac{n}{5}\right) = \frac{n}{30}$ .

If  $\frac{5n}{c} < 24 < \frac{5n}{b} \leq 25$ , then let  $m = 24$  and we have done. Otherwise, we have

$$\begin{aligned} \frac{27}{2} &< \frac{3n}{c} \leq 14 < \frac{3n}{b} \leq 15, \\ 18 &< \frac{4n}{c} \leq 19 < \frac{4n}{b} \leq 20, \\ \frac{45}{2} &< \frac{5n}{c} \leq 23 < \frac{5n}{b} < 24, \end{aligned}$$

there exists at least one of integers 14, 19, 23 is co-prime to  $n$ . Then  $\text{ind}(S) = 1$ .

**Case 7.**  $t = 6$ . We have  $5 < \frac{n}{c} < \frac{n}{b} \leq \frac{11}{2}$  and  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{n}{5} - \frac{2n}{11}\right) < \frac{n}{36}$ .

We also have  $15 < \frac{3n}{c} < 16 < \frac{3n}{b} \leq \frac{33}{2}$ , let  $m = 16$ . Then  $ma < n$  and  $\text{ind}(S) = 1$ .

**Case 8.**  $t = 7$ . We have  $\frac{11}{2} < \frac{n}{c} < \frac{n}{b} < 6$  and  $a \leq \frac{3(a-e)}{2} = \frac{3(c-b)}{2} < \frac{3}{2} \times \left(\frac{2n}{11} - \frac{n}{6}\right) = \frac{n}{44}$ .

We also have

$$\begin{aligned} \frac{33}{2} &< \frac{3n}{c} \leq 17 < \frac{3n}{b} < 18, \\ 22 &< \frac{4n}{c} \leq 23 < \frac{4n}{b} < 24, \\ \frac{55}{2} &< \frac{5n}{c} \leq m_1 < \frac{5n}{b} < 30, \end{aligned}$$

where  $m_1 \in \{28, 29\}$ , and there exists at least one of integers 17, 23,  $m_1$  is co-prime to  $n$ . Then  $\text{ind}(S) = 1$ .  $\square$

**Lemma 5.6.** *Let  $e, a, b, c$  be parameters listed in Proposition 2.5. If  $n = 5^\alpha 7^\beta$  and  $\frac{3n}{8} < b < c < \frac{11n}{23}$ , then  $\frac{n}{9} \geq a$ .*

*Proof.* **case 1.**  $e = p^{i_0}$ :

If  $e = 5$  or  $e = 7$ , then  $n > \frac{1000}{7}e \geq 142e$ . If  $e \geq 25$ , then  $n \geq 5p^{i_0}q^{j_0} \geq 5e^2 \geq 125e$ .

**case 2.**  $e = q^{j_0}$ :

If  $e = 7$ ,  $n > 142e$ . Clearly,  $e$  can't equal to 25, otherwise we can't find suitable  $p^{i_0}$ . When  $e = 49$ , we have  $p^{i_0} = 25$  and  $n \geq 5p^{i_0}e = 125e$ . If  $e \geq 125$ , we have  $p^{i_0} > \frac{e}{3}$  and  $n \geq 5p^{i_0} > 208$ .

Both of the above cases, we have  $n \geq 125e$ . If  $\frac{n}{9} < a$ , then

$$\frac{n}{9} < a < \frac{11n - p^{i_0}}{23} - \frac{3n + q^{j_0}}{8} + e \leq \frac{19n + 169e}{184},$$

hence we have  $n < 117e$ , which contradicts to  $n \geq 125e$ .

**case 3.**  $e = 2q^{j_0}$ . Clearly,  $e \notin \{10, 50\}$ .

*subcase 3.1.*  $e = 14$ . If  $n \geq 5^4 7$ , then  $n \geq \frac{5^4}{2}e > 322e$ . The proof is similar to above.

Otherwise  $n = 5^2 7^2$ . Then  $a \in \{2 \times (2t + 1) \times 7, n - \frac{n}{7} + 10\}$ . Since  $5|(2t + 1 - 1)$ , we have  $t \geq 5$ . Moreover,  $n - \frac{n}{7} + 10 = 75 \times 14 + 10$ . So  $a \geq 11e$ . Then we have

$$a \leq \frac{11}{10}(a - e) < \frac{11}{10} \left( \frac{11n}{23} - \frac{3n}{8} \right) = \frac{201n}{1840} = \frac{n}{9} \times \frac{1809}{1840} < \frac{n}{9}.$$

*subcase 3.2.*  $e = 98$ . The proof is similar to subcase 3.1.

*subcase 3.3.*  $e \geq 250$ , we have  $n > 312e$  and the proof is similar to Case 1 and Case 2.  $\square$

**Lemma 5.7.** *Let  $k_1 = 2$ ,  $4 < \frac{2n}{c} \leq 5 < \frac{2n}{b} < 6$  and  $a \leq \frac{b}{2}$ . If the assumption is as in Proposition 2.8, then  $\text{ind}(S) = 1$ .*

*Proof.* Then  $4 < \frac{2n}{c} \leq 5 < \frac{2n}{b} < 6$ . If  $6a > n$ , then  $2n < 6c, 6b < 3n, n < 6a < 2n, 6e < 2a < n$ , we have  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n = 3n$ .

If  $6a < n$  and  $\gcd(n, 5) = 1$ , let  $m = 5$ , we have  $\frac{2n}{c} \leq 5 < \frac{2n}{b}$ , then  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n = n$ .

Next we assume that  $5|n$  and  $6a < n$ .

**Case 1.**  $7 < \frac{3n}{c} < 8 < \frac{3n}{b} < 9$ . If  $8a < n$ , let  $m = 8$ , we have done.

If  $8a > n$ , let  $m = 9$ . Then  $3n < 9b < 9c < \frac{27n}{7} < 4n$  and  $9e < 3a < n$ . We have  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n \geq 3n$ , hence  $\text{ind}(S) = 1$ .

**Case 2.**  $6 < \frac{3n}{c} < 7 < \frac{3n}{b} < 8$  and  $\gcd(n, 7) = 1$ . We have  $a < \frac{3}{2} \left( \frac{n}{2} - \frac{3n}{8} \right) < \frac{n}{5}$ .

If  $7a < n$ , let  $m = 7$ , we have done.

If  $7a > n$ , let  $m = 14$ . Then  $6n < 14c < 7n, 5n < \frac{40b}{3} < 14b < 6n$  and  $14e < 5a < n$ . We have  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n \geq 3n$ , hence  $\text{ind}(S) = 1$ .

**Case 3.**  $6 < \frac{3n}{c} < 7 < \frac{3n}{b} < 8$  and  $\gcd(n, 7) > 1$ .

Note that  $8 < \frac{4n}{c} \leq 10 < \frac{4n}{b} < 12$ .

If  $9 < \frac{4n}{c} \leq 10 < \frac{4n}{b} < 12$ , we have  $\frac{5n}{c} \leq \frac{35}{3} < 12 = \frac{10 \times 6}{5} < \frac{5n}{b}$  and

$$a < \begin{cases} \left(\frac{4n}{9} - \frac{3n}{8} + \frac{n}{75}\right) < \frac{n}{12}, & e = p^{i_0}, \\ \frac{6}{5} \times \left(\frac{4n}{9} - \frac{3n}{8}\right) = \frac{n}{12}, & e \neq p^{i_0}, \end{cases}$$

let  $m = 12$  and  $k = 5$ , then we have done.

If  $8 < \frac{4n}{c} < 9 < 10 < \frac{4n}{b}$ , then  $\frac{3n}{8} < b < \frac{2n}{5} < \frac{4n}{9} < c$  and

$$8n + \frac{n}{2} < \frac{69n}{8} < 23b < \frac{46n}{5} < 9n + \frac{n}{2} < 10n < \frac{92n}{9} < 23c < \frac{23n}{2} = 11n + \frac{n}{2}.$$

Note that  $a = c - b + e \leq \frac{n-p^{i_0}}{2} - \frac{3n+p^{i_0}}{8} + e = \frac{n-5p^{i_0}}{8} + e$ . If  $a > \frac{n}{8}$ , then let  $M = 12$ . We obtain that  $|Me|_n < \frac{n}{2}$ ,  $|Mb|_n > \frac{n}{2}$  and  $|Ma|_n > \frac{n}{2}$  since

$$\frac{3n}{2} < Ma \leq \frac{3n}{2} + 12e - \frac{15p^{i_0}}{2}$$

and

$$12e - \frac{15p^{i_0}}{2} \leq \begin{cases} 9p^{i_0} < \frac{3n}{25} < \frac{n}{2}, & e = p^{i_0}, \\ 12e \leq 2a < \frac{n}{2}, & e \neq p^{i_0}, \end{cases}$$

and we have done.

If  $9a < n$ , let  $m = 9, k = 4$ . Then  $\text{ind}(S) = 1$ .

Then we assume that  $\frac{n}{9} < a < \frac{n}{8}$ , and thus

$$9n = \frac{3n}{8} \times 24 < 24b < 24 \times \frac{2n}{5} < 10n < 24 \times \frac{4n}{9} < 24c < 12n.$$

By Lemma 5.6, we have  $23c > 11n$ . Then  $|23c|_n < \frac{n}{2}$ . By Proposition 2.5, we have  $|23e|_n = 23e < \frac{n}{2}$ . We also have  $\frac{5n}{2} < \frac{23n}{9} < 23a < \frac{23n}{8} < 3n$ , hence  $|23a|_n > \frac{n}{2}$ . Then we have  $\text{ind}(S) = 1$ .

**Case 4.**  $6 < \frac{3n}{c} \leq 7 < 8 < \frac{3n}{b} < 9$ . We distinguish three subcases.

*Subcase 4.1.*  $\gcd(n, 77) = 1$ .

We may assume that  $a > \frac{n}{7}$  (for otherwise, if let  $m = 7$  and  $k = 3$ , we have  $ma < n$ , so the lemma follows from Lemma 2.3 (1)). Hence  $n < 11a < 2n$ . Also, we have that  $3n < \frac{11n}{3} < 11b < \frac{33n}{8} < 5n$  and  $4n < \frac{33n}{7} < 11c < \frac{11n}{2} < 6n$ .

If  $11b < 4n$  and  $11c > 5n$ , we have  $|11e|_n + |11c|_n + |11(n-b)|_n + |11(n-a)|_n = 11e + (11c - 5n) + (4n - 11b) + (2n - 11a) = n$  and thus  $\text{ind}(S) = 1$ .

If  $11b > 4n$  and  $11c < 5n$ , we have  $|11e|_n + |11c|_n + |11(n-b)|_n + |11(n-a)|_n = 11e + (11c - 4n) + (5n - 11b) + (2n - 11a) = 3n$  and thus  $\text{ind}(S) = 1$  (by Remark 2.1 (2)).

If  $11b < 4n$  and  $11c < 5n$ , then we have either  $\frac{n}{7} < a = c - b + e \leq \frac{5n}{11} - \frac{n}{3} + e$ , which implies that  $n < 47e$ , or  $\frac{n}{7} < a \leq \frac{25}{24}(a - e) = \frac{25}{24}(c - b) < \frac{25n}{198} < \frac{25n}{175} = \frac{n}{7}$ . By Lemma 3.1, both of them lead to a contradiction.

If  $11b > 4n$  and  $11c > 5n$ , then either  $\frac{n}{7} < a = c - b + e \leq \frac{n-e}{2} - \frac{4n-e}{11} + e$ , which implies that  $n < 63e$ , or  $\frac{n}{7} < a \leq \frac{25}{24}(a - e) = \frac{25}{24}(c - b) < \frac{25n}{176} < \frac{25n}{175} = \frac{n}{7}$ . By Lemma 3.1, both of them lead to a contradiction.

*Subcase 4.2.*  $55|n$ .



As in Subcase 4.1, we may assume that  $a > \frac{n}{7}$ . Then

$$\frac{3n}{2} < \frac{13n}{7} < 13a < \frac{13n}{6} < \frac{5n}{2} < 4n < \frac{13n}{3} < 13b < \frac{39n}{8} < 5n < \frac{11n}{2} < \frac{39n}{7} < 13c < \frac{13n}{2}.$$

If  $13c < 6n$ , then  $\frac{n}{7} < a = c - b + e \leq \frac{6n}{13} - \frac{n}{3} + e$ , so  $n < 69e$ , yielding a contradiction by Lemma 3.1. Hence we must have that  $13c > 6n$ , and then  $|13c|_n < \frac{n}{2}$ . Moreover, we have  $13e < \frac{n}{2}$  by Lemma 3.1.

If  $13a < 2n$  or  $13b > \frac{9n}{2}$ , then  $|13a|_n > \frac{n}{2}$  or  $|13b|_n > \frac{n}{2}$ . Since  $\gcd(n, 13) = 1$ , the lemma follows from Lemma 2.3 (2) with  $M = 13$ . Next we assume that  $13a > 2n$  and  $13b < \frac{9n}{2}$ . Then  $\frac{2n}{13} < a < \frac{n}{6}$  and  $\frac{n}{3} < b < \frac{9n}{26}$ . Therefore,

$$\frac{5n}{2} < \frac{34n}{13} < 17a < \frac{17n}{6} < 3n < \frac{11n}{2} < \frac{17n}{3} < 17b < \frac{153n}{26} < 6n.$$

We infer that  $|17a|_n > \frac{n}{2}$  and  $|17b|_n > \frac{n}{2}$ . Since  $\gcd(n, 17) = 1$  and  $17e < \frac{n}{2}$ , the lemma follows from Lemma 2.3 (2) with  $M = 17$ .

*Subcase 4.3.*  $35|n$ . As in Subcase 4.1, we may assume that  $a > \frac{n}{8}$ . By using a similar argument in Subcase 4.2 and Lemma 3.1, we can complete the proof with  $M = 11$  or  $M = 13$ .  $\square$

**Lemma 5.8.** *If the assumption is as in Proposition 2.8 and  $k_1 = 2$ , then  $\text{ind}(S) = 1$ .*

*Proof.* **Case 1.**  $5 < \frac{n}{c} < \frac{n}{b} < 6$ . Then  $10 < \frac{2n}{c} < 11 < \frac{2n}{b} < 12$ . If  $\gcd(n, 11) = 1$ , then  $a < \frac{3}{2}(a - e) = \frac{3}{2}(c - b) < \frac{3}{2}(\frac{n}{5} - \frac{n}{6}) = \frac{n}{20}$ ,  $11a < n$ . Let  $m = 11$ ,  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n = n$ .

Since  $15 < \frac{3n}{c} < \frac{33}{2} < \frac{3n}{b} < 18$ , if  $\frac{3n}{c} < 16$ , then we have done. If  $16 < \frac{3n}{c} < 17 < \frac{3n}{b} < 18$  and  $\gcd(17, n) = 1$ , let  $m = 17$ , then  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n = m$ .

If  $16 < \frac{3n}{c} < \frac{3n}{b} < 17$ , then  $a < \frac{3}{2}(\frac{3n}{16} - \frac{3n}{17}) = \frac{3n}{272} < \frac{n}{90} < \frac{b}{15}$ , a contradiction.

Now let  $11|n, 17|n$  and  $\frac{n}{c} < \frac{11}{2} < \frac{17}{3} < \frac{n}{b}$ . Then  $\frac{5n}{c} < \frac{55}{2} < 28 < \frac{85}{3} < \frac{5n}{b}$  and  $a < \frac{3}{2}(\frac{3n}{16} - \frac{n}{6}) = \frac{n}{32}$ . Let  $m = 28$ , we have  $\gcd(n, m) = 1$  and  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n = n$ .

**Case 2.**  $4 < \frac{n}{c} < \frac{n}{b} \leq 5$ . Then  $8 < \frac{2n}{c} < 9 < \frac{2n}{b} \leq 10$  and  $a < \frac{3}{2}(a - e) = \frac{3}{2}(c - b) < \frac{3}{2}(\frac{n}{4} - \frac{n}{5}) = \frac{3n}{40}$ . Let  $m = 9$ , we have  $\gcd(n, m) = 1$  and  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n = n$ .

**Case 3.**  $3 < \frac{n}{c} < \frac{n}{b} < 4$ . Then  $6 < \frac{2n}{c} < 7 < \frac{2n}{b} \leq 8$  and  $a < \frac{3}{2}(a - e) = \frac{3}{2}(c - b) < \frac{3}{2}(\frac{n}{3} - \frac{n}{4}) = \frac{n}{8}$ . If  $\gcd(n, 7) = 1$ , let  $m = 7$ , we have  $\gcd(n, m) = 1$  and  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n = n$ .

If  $7|n$ , we divide the proof into the following four subcases.

*Subcase 3.1* If  $\frac{n}{c} < \frac{10}{3} < \frac{11}{3} < \frac{n}{b}$ . Then at least one of 10, 11 is co-prime to  $n$ . Let  $m \in \{10, 11\}$  be such that  $\gcd(m, n) = 1$ . If  $ma < n$ , then  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n = n$ .

If  $ma > n$ , then  $3n < 12c < 4n$ ,  $3n < 12b < 4n$ ,  $n < 12a < 2n$ ,  $12e < 4a < n$ , we have  $|12e|_n + |12c|_n + |12(n - b)|_n + |12(n - a)|_n = 3n$ .

*Subcase 3.2* If  $\frac{10}{3} < \frac{n}{c} < \frac{n}{b} < \frac{15}{4}$ . Then  $a < \frac{3}{2}(\frac{3n}{10} - \frac{4n}{15}) = \frac{n}{30} < \frac{b}{8}$ , a contradiction.

*Subcase 3.3* If  $\frac{10}{3} < \frac{n}{c} < \frac{15}{4} < \frac{n}{b}$ . Then  $a < \frac{3}{2}(\frac{3n}{10} - \frac{n}{4}) = \frac{3n}{40}$ .

We have  $\frac{4n}{c} < 14 < 15 < \frac{4n}{b}$ ,  $\frac{6n}{c} < 21 < 22 < \frac{6n}{b}$ .

If  $15a > n$ , we have  $4n < 16c < 5n$ ,  $4n < 16b < 5n$ ,  $n < 16a < 2n$ ,  $16e < n$ , and let  $m = 16$ ,  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n = 3n$ .

If  $15a < n$ ,  $\gcd(n, 15) = 1$ , let  $m = 15$  we have  $|me|_n + |mc|_n + |m(n - b)|_n + |m(n - a)|_n = n$ .

If  $22a > n$ , let  $m = 23$ ,  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n = 3n$ . If  $22a < n$ , let  $m = 22$ ,  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n = n$ .

*Subcase 3.4.* If  $3 < \frac{n}{c} < \frac{10}{3} < \frac{n}{b} < \frac{11}{3}$ . Then  $a < \frac{2n}{33}$ . If  $\gcd(n, 10) = 1$ , let  $m = 10$ , we have  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n = n$ .

Let  $5|n$ . If  $16c > 5n$ , since  $4n < 16b < 5n$ ,  $16e < 16a < n$ , let  $m = 16$ , we have  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n = n$ .

If  $16c < 5n$  and  $17b < 5n$ , then  $a < \frac{n}{24}$ , let  $m = 17$ , we have  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n = n$ . If  $16c < 5n$  and  $17b > 5n$ , then  $a < \frac{n}{51} < \frac{b}{15}$ , which contradicts to  $8a > b$ .

**Case 4.**  $2 < \frac{n}{c} < \frac{n}{b} < 3$ .

Since  $k_1 = 2$ , we have  $4 < \frac{2n}{c} \leq 5 < \frac{2n}{b} < 6$ , so  $m_1 = 5$ . Since  $\gcd(n, m_1) > 1$ , we have  $5|n$ . The result now follows from Lemma 5.6.  $\square$

Now Proposition 2.8 follows immediately from Lemmas 5.1-5.5 and Lemma 5.8.

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## REFERENCES

- [1] S.T. Chapman, M. Freeze, and W.W Smith, *Minimal zero sequences and the strong Davenport constant*, Discrete Math. 203(1999), 271-277.
- [2] S.T. Chapman, and W.W Smith, *A characterization of minimal zero-sequences of index one in finite cyclic groups*, Integers 5(1)(2005), Paper A27, 5p.
- [3] W. Gao, *Zero sums in finite cyclic groups*, Integers 0 (2000), Paper A14, 9p.
- [4] W. Gao and A. Geroldinger, *On products of  $k$  atoms*, Monatsh. Math. 156 (2009), 141-157.
- [5] W. Gao, Y. Li, J. Peng, P. Plyley and G. Wang *On the index of sequences over cyclic groups* (English), Acta Arith. 148, No. 2, (2011) 119-134.
- [6] A. Geroldinger, *On non-unique factorizations into irreducible elements. II*, Number Theory, Vol II Budapest 1987, Colloquia Mathematica Societatis Janos Bolyai, vol. 51, North Holland, 1990, 723-757.
- [7] A. Geroldinger, *Additive group theory and non-unique factorizations*, Combinatorial Number Theory and Additive Group Theory (A. Geroldinger and I. Ruzsa, eds.), Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2009, pp. 1-86.
- [8] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations*. Algebraic, Combinatorial and Analytic Theory, Pure and Applied Mathematics, Vol. 278, Chapman & Hall/CRC, 2006.
- [9] D. Kleitman and P. Lemke, *An addition theorem on the integers modulo  $n$* , J. Number Theory 31(1989), 335-345.
- [10] Y. Li and J. Peng, *Minimal zero-sum sequences of length five over finite cyclic groups*, Ars Combinatoria, to appear.
- [11] Y. Li and J. Peng, *Minimal zero-sum sequences of length four over finite cyclic groups II*, International Journal of Number Theory 09 (2013), 845-866.
- [12] Y. Li, C. Plyley, P. Yuan and X. Zeng, *Minimal zero sum sequences of length four over finite cyclic groups*, Journal of Number Theory. 130 (2010), 2033-2048.
- [13] V. Ponomarenko, *Minimal zero sequences of finite cyclic groups*, Integers 4(2004), Paper A24, 6p.
- [14] S. Savchev and F. Chen, *Long zero-free sequences in finite cyclic groups*, Discrete Math. 307 (2007), 2671-2679.
- [15] X. Xia and P. Yuan, *Indexes of insplitable minimal zero-sum sequences of length  $l(C_n) - 1$* , Discrete Math. 310 (2010), 1127-1133.

- [16] P. Yuan, *On the index of minimal zero-sum sequences over finite cyclic groups*, J. Combin. Theory Ser. A114(2007), 1545-1551.
- [17] P. Yuan and X. Zeng, *Indexes of long zero-sum free sequences over cyclic groups*, Eur. J. Comb. 32(2011), 1213-1221.
- [18] D. J. Gryniewicz, *Structural Additive Theory, Developments in Mathematics*, to appear, Springer, 2013.
- [19] L. Xia, *On the index-conjecture on length four minimal zero-sum sequences*, International Journal of Number Theory, to appear, DOI: 10.1142/S1793042113500401.